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ON THE STRONG LAW OF LARGE NUMBERS FOR DEPENDENT RANDOM VARIABLES

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ABSTRACT

Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of random variables with partial sums S_n, and let $\mathcal{F}_n$ be the σ-algebra generated by $X_1, \cdots, X_n$. Let f be a function from R to R and suppose $E(X_{n+1} \mid \mathcal{F}_n) = f(S_n/n)$. Under conditions of f and moment conditions on the X_n's, we show that S_n/n converges a.e. (almost everywhere). We give several applications of this result.

1. Introduction

Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of random variables and for each n let $S_n = \sum_{i=1}^{n} X_i$. Let $\mathcal{F}_n$ be the σ-algebra generated by $X_1, \cdots, X_n$. If the random variables X_n are independent with $E(X_n) = 0$ and if $\sum_{i=1}^{\infty} E(X_i^2)/i^2 < \infty$ then the classical strong law of large numbers states that $\lim_n S_n/n = 0$ a.e. (almost everywhere). More generally the same conclusion holds if $\sum_{i=1}^{\infty} E(X_i^2)/i^2 < \infty$ and $E(X_{n+1} \mid \mathcal{F}_n) = 0$ a.e. for all n. On the other hand, if $E(X_{n+1} \mid \mathcal{F}_n) = S_n/n$ a.e. for all n, then S_n/n is a martingale and if $\sum_{i=1}^{\infty} E(X_i^2)/i^2 < \infty$ we again have the a.e. convergence of S_n/n.

In this note, we generalize these two cases by assuming $E(X_{n+1} \mid \mathcal{F}_n) = f(S_n/n)$ where f is a function from R to R. In Section 2, we show that under appropriate conditions on f we obtain the a.e. convergence of S_n/n. In Section 3, we give several applications of this result.

2. The main theorem

We assume throughout that $\sum_{i=1}^{\infty} E(X_i^2)/i^2 < \infty$. Then we have

THEOREM 1. *Let $\{X_n\}_{n=1}^{\infty}$ be random variables such that $E(X_{n+1} \mid \mathcal{F}_n) = f(S_n/n)$* a.e. *for all n, where f is a function from R to R satisfying*:

(2.1) *There exist numbers a and b such that $|f(x)| \leqq a + b|x|$.*

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(2.2) *There exists a number* $M > 0$ *such that* $f(x) \geqq x$ *for* $x < -M$, *and* $f(x) \leqq x$ *for* $x > M$.

(2.3) *Every open subinterval* I *of* $[-M, M]$ *contains a further subinterval* I' *such that* $f(x) \geqq x$ *for all* $x \in I'$ *or* $f(x) \leqq x$ *for all* $x \in I'$.

Then S_n/n *converges* a.e.

PROOF. Let $M_n = \Sigma_{i=1}^n [X_{i+1}/i+1 - E(X_{i+1}/i+1 \mid \mathcal{F}_i)]$. Then M_n is an L_2 martingale and therefore converges a.e. Now

$$E(S_{n+1}/n+1 \mid \mathcal{F}_n) = \frac{1}{n+1}[S_n + f(S_n/n)] \qquad \text{a.e.}$$

and hence

$$S_{n+1}/n+1 - E(S_{n+1}/n+1 \mid \mathcal{F}_n) = \frac{1}{n+1}[X_{n+1} - f(S_n/n)].$$

Therefore

$$\sum_{i=1}^n [S_{i+1}/i+1 - E(S_{i+1}/i+1 \mid \mathcal{F}_i)]$$

$$= \sum_{i=1}^n [(S_{i+1}/i+1 - S_i/i) - (E(S_{i+1}/i+1 \mid \mathcal{F}_i) - S_i/i)]$$

$$= (S_{n+1}/n+1 - X_1) - \sum_{i=1}^n \frac{1}{i+1}[f(S_i/i) - S_i/i].$$

But

$$M_n = \sum_{i=1}^n \frac{1}{i+1}[X_{i+1} - f(S_i/i)] = \sum_{i=1}^n [S_{i+1}/i+1 - E(S_{i+1}/i+1 \mid \mathcal{F}_i)],$$

and therefore

$$(2.4) \qquad (S_{n+1}/n+1 - X_1) - \sum_{i=1}^n \frac{1}{i+1}[f(S_i/i) - S_i/i] \quad \text{converges a.e.}$$

From now on, we shall only consider sample sequences for which this convergence holds. If $\{X_n\}_{n=1}^\infty$ is such a sequence we must have $\underline{\lim}\, S_n/n < \infty$. For otherwise we would have $S_n/n \geqq M$ for $n \geqq n_0$ and hence $f(S_n/n) - S_n/n \leqq 0$ a.e. by (2.2). But this contradicts (2.4). Similarly $\overline{\lim}\, S_n/n > -\infty$.

Suppose now that $\underline{\lim}\, S_n/n < \overline{\lim}\, S_n/n$ on a set of sequences of positive probability. We shall show that for any such sequence we must have $-M \leqq \underline{\lim}\, S_n/n < \overline{\lim}\, S_n/n \leqq M$. Otherwise assume e.g. that $M < \overline{\lim}\, S_n/n$. Then we can find numbers α and β such that $\max[\underline{\lim}\, S_n/n, M] < \alpha < \beta < \overline{\lim}\, S_n/n$. Therefore there are infinitely many pairs of integers (n, m) with $n < m$ and such that

$S_n/n < \alpha < \beta < S_m/m$ and $\alpha \leqq S_j/j \leqq \beta$ for $n < j < m$. From (2.4) we may choose such a pair so large that

$$(S_m/m - S_n/n) - \sum_{i=n}^{m-1} \frac{1}{i+1}[f(S_i/i) - S_i/i] < (\beta - \alpha)/10.$$

But for $n < i < m$ we have $f(S_i/i) - S_i/i \leqq 0$ by (2.2) and therefore

$$(S_m/m - S_n/n) - \frac{1}{n+1}[f(S_n/n) - S_n/n] < (\beta - \alpha)/10.$$

Now if $S_n/n \geqq M$, then also $f(S_n/n) - S_n/n \leqq 0$ and we obtain $(S_m/m - S_n/n) < (\beta - \alpha)/10$ which is impossible since clearly $(S_m/m - S_n/n) > (\beta - \alpha)$.

To handle the case $S_n/n < M$ we use the estimate

$$\begin{aligned}
(S_m/m - S_n/n) &< (\beta - \alpha)/10 + \frac{1}{n+1}[f(S_n/n) - S_n/n] \\
&\leqq (\beta - \alpha)/10 + \frac{1}{n+1}[|f(S_n/n)| + |S_n/n|] \\
&\leqq (\beta - \alpha)/10 + \frac{1}{n+1}[a + (b+1)|S_n/n|] \quad \text{(by (2.1))} \\
&\leqq (\beta - \alpha)/10 + \frac{1}{n+1}[a + (b+1)|M - S_n/n| + (b+1)|M|] \\
&\leqq (\beta - \alpha)/10 + \frac{1}{n+1}[a + (b+1)(S_m/m - S_n/n) + (b+1)M]
\end{aligned}$$

since $S_n/n < M < S_m/m$ and $M > 0$. Finally we combine these inequalities to obtain

$$(S_m/m - S_n/n)\left(1 - \frac{b+1}{n+1}\right) \leqq (\beta - \alpha)/10 + \frac{1}{n+1}[a + (b+1)M]$$

which again leads to a contradiction for n sufficiently large. Therefore $\overline{\lim}\, S_n/n \leqq M$ and similarly $\underline{\lim}\, S_n/n \geqq -M$.

Now suppose that $\{X_n\}_{n=1}^{\infty}$ is a sample sequence for which $-M \leqq \underline{\lim}\, S_n/n < \overline{\lim}\, S_n/n \leqq M$. Then S_n/n is bounded and therefore by (2.1) $f(S_n/n)$ is also bounded. Consequently we have

$$\lim_n \frac{1}{n+1}[f(S_n/n) - S_n/n] = 0.$$

But $(S_{n+1}/n+1 - S_n/n) - (1/(n+1))[f(S_n/n) - S_n/n]$ is the term of a convergent series and therefore we also have $\lim_n (S_{n+1}/n+1 - S_n/n) = 0$. Let I be an open subinterval of $[\underline{\lim}\, S_n/n, \overline{\lim}\, S_n/n]$ and let I' be the subinterval of I guaranteed by (2.3) and let x_1 and x_2 be the left and right endpoints of I'. Assume e.g. that $f(x) \leqq x$ for $x_1 < x < x_2$. Choose x_1' and x_2' such that $x_1 < x_1' < x_2' < x_2$. Again we may find infinitely many pairs of integers (n, m) with $n < m$ and $x_1 < S_n/n < x_1'$, $x_2' < S_m/m < x_2$ and $x_1' \leqq S_j/j \leqq x_2'$ for $n < j < m$.

But then we would have

$$(S_m/m - S_n/n) - \sum_{i=n}^{m-1} \frac{1}{i+1} [f(S_i/i) - S_i/i] \geqq (S_m/m - S_n/n) > x_2' - x_1' > 0$$

which violates (2.4). This proves the theorem.

Let S be the a.e. limit of S_n/n. While a detailed analysis of the support of S seems difficult, we have a partial result.

THEOREM 2. *If f is continuous then $P\{S = f(S)\} = 1$.*

The proof is similar to that of Theorem 1 and will be omitted. Further results on the support of S when f maps the unit interval into itself may be found in [1].

It is worth nothing that something like condition (2.1) on f is needed. For suppose $f(x) = -x|x|^\delta$ with $\delta > 0$. Consider the nonrandom scheme $X_{n+1} = f(S_n/n)$, with X_1 arbitrary. Let $a_n = |S_n/n|$. Then it is easily verified that

$$a_{n+1} = \left| \frac{a_n^{1+\delta} - na_n}{n+1} \right|,$$

and we have $a_{n+1} \geqq 2a_n$ provided $a_n \geqq (3n+2)^{1/\delta}$. We can accomplish this by choosing X_1 so that $2^{n-1}|X_1| \geqq (3n+2)^{1/\delta}$ for all n. In that case $\lim_n a_n = \infty$.

3. Application

We know of two areas in which these results may be applied. Consider an urn which has initially n_0 balls of which a proportion S_{n_0}/n_0 is red and the remaining ones are black. Let f be a function mapping the unit interval into itself. A red ball is added to the urn with probability $f(S_{n_0}/n_0)$ or a black ball with probability $1 - f(S_{n_0}/n_0)$. The new proportion of red balls is $S_{n_0+1}/n_0 + 1$ and this process is repeated indefinitely. The problem is to show that S_n/n converges a.e.

In [1] our results were obtained under the assumptions that X_n is 0 or 1 for all n, and that f maps the unit interval into itself. However our methods are completely different than those used in [1].

Now consider the problem of stochastic approximation. Let g be an unknown function mapping R into R, and assume for each $x \in R$ one may perform an experiment described by a random variable Y_x such that $E(Y_x) = g(x)$. Assume there exists θ such that $g(x) < 0$ for $x < \theta$ and $g(x) > 0$ for $x > \theta$. The object is to estimate θ.

Now choose Z_1 arbitrarily and define recursively $Z_{n+1} = Z_n - \alpha_n Y_n$, where $\{\alpha_n\}_{n=1}^{\infty}$ is a sequence of positive numbers such that $\lim_n \alpha_n = 0$ and $\Sigma_{n=1}^{\infty} \alpha_n = \infty$, and Y_n has the distribution of Y_{Z_n}. Under various assumptions on g and the moments of the family $\{Y_r\}$ one then obtains that $\lim_n Z_n = \theta$ a.e.

Now in our result write $S_{n+1}/n + 1 = S_n/n + X_{n+1}/n + 1$ and also write $X_{n+1} = f(S_n/n) - U_n$, where $U_n = f(S_n/n) - X_{n+1}$. Finally let $g(x) = x - f(x)$. Then we obtain

$$S_{n+1}/n + 1 = S_n/n - \frac{1}{n+1}[g(S_n/n) + U_n].$$

If we make the identification $S_n/n = Z_n$, and $g(S_n/n) + U_n = Y_n$, then we see that our scheme is in fact a stochastic approximation scheme.

However there are some major differences in our assumptions and correspondingly some differences in the conclusions. For example in stochastic approximation, one always assumes that g has a unique root at θ, and that in any closed interval not containing θ, g is bounded away from zero. We only assume the existence of $M > 0$ such that $g(x) \leqq 0$ for $x < -M$, and $g(x) \geqq 0$ for $x > M$. On the other hand we only obtain a.e. convergence of the scheme, and cannot identify the limit. Also our second moment assumptions are somewhat weaker than those usually made in stochastic approximation.

Reference

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